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ON THE STRUCTURE OF THE VELOCITY FIELD IN THE VICINITY OF A CAUSTIC

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A problem of reflection of a short wave from a caustic is studied. Approximate equations are derived in which the principal nonlinear term is retained. A solution is obtained for an arbitrary incident wave in the linear approximation, and this makes possible a description of the reflected wave. A qualitative dededuction is made about the influence of the wavelength of the arriving wave containing a strong discontinuity, on the amplitude of the reflected signal.

Let us consider a plane problem of propagation of a short sound wave in an inhomogeneous medium. We shall assume that the wavelength λ is much smaller than the characteristic dimension L of the inhomogeneous medium. Under these conditions, the approximation of geometrical acoustics holds. Let the rays along which the signal propagates, have an envelope. This envelope shall be called the caustic. The approximation of geometrical acoustics breaks down in the neighborhood of this caustic. Using the method of matching the inner and outer asymptotic expansions, we can construct a solution of the wave equation near the caustic [1,2[. The thickness of the boundary layer l will have the order of $\lambda (L/\lambda)^{1/3}$. If a strong discontinuity is present in the wave arriving at the caustic, then the solution describing the reflected wave will contain a logarithmic singularity [3] caused by the fact that the wave equation itself is a linear approximation of the gasdynamic equations. Nevertheless, in spite of this, the linear solution helps to elucidate the qualitative pattern of the flow near the caustic and to extract useful information concerning the character of reflection of the short waves from the caustic.

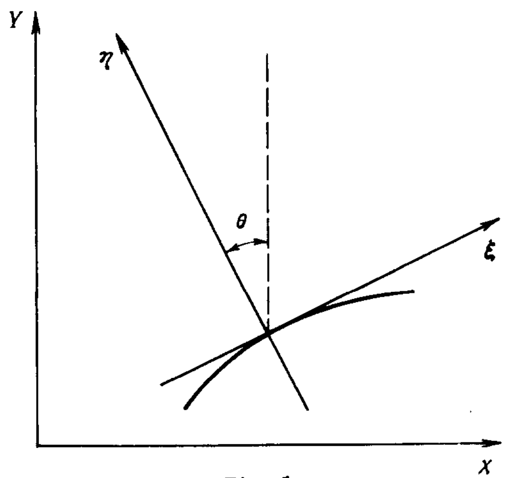

Fig. 1

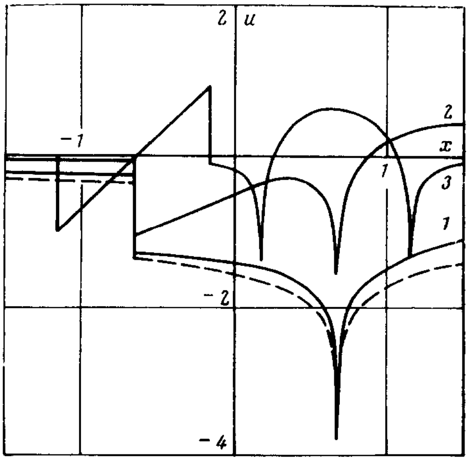

Fig. 2

Let the caustic be defined by the following equations in the Cartesian X, Y, t -coordinates:

$$X = f(t), \quad Y = g(t)$$

We pass to the intrinsic coordinate system moving along the caustic together with the wave (Fig. 1)

$$\xi = (X - f(t)) \cos\theta + (Y - g(t)) \sin\theta$$
$$\eta = -(X - f(t)) \sin\theta + (Y - g(t)) \cos\theta$$

where θ is the angle between the normal to the caustic and the ordinate. The ξ -axis is oriented along the caustic in the direction opposing the motion of the coordinate system, and the η -axis is directed along the normal to the caustic, towards the arriving wave.

The gasdynamic equations in this coordinate system have the form (θ' denotes the angular rate of rotation of the coordinate system, and a_c is the speed of sound at the caustic)

$$\frac{\partial\rho}{\partial t} + (a_c + u + \eta\theta')\frac{\partial\rho}{\partial\xi} + (v - \xi\theta')\frac{\partial\rho}{\partial\eta} + \rho\left(\frac{\partial u}{\partial\xi} + \frac{\partial v}{\partial\eta}\right) = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} - v\theta' + (a_c + u + \eta\theta')\frac{\partial u}{\partial\xi} + (v - \xi\theta')\frac{\partial u}{\partial\eta} + \frac{1}{\rho}\frac{\partial p}{\partial\xi} = 0$$

$$\frac{\partial v}{\partial t} + u\theta' + (a_c + u + \eta\theta')\frac{\partial v}{\partial\xi} + (v - \xi\theta')\frac{\partial v}{\partial\eta} + \frac{1}{\rho}\frac{\partial p}{\partial\eta} = 0$$

$$\frac{\partial S}{\partial t} + (a_c + u + \eta\theta')\frac{\partial S}{\partial\xi} + (v - \xi\theta')\frac{\partial S}{\partial\eta} = 0$$

Since the coordinate system moves together with the wave, therefore the time-dependent changes caused by dispersion of the rays will be proportional to a_c / L, while the variations in ξ and η will be proportional to $1/\lambda$ and $1/l$ respectively. Since $\lambda / L \ll 1$ and $l / L = (\lambda / L)^{2/3} \ll 1$, the motion is steady in the first approximation. We also assume that the perturbations in all the quantities are small, namely $u \ll a_c$, $v \ll a_c$, $p - p_0 \ll p_0$, $\rho - \rho_0 \ll \rho_0$. Then, instead of (1) we obtain, in the basic order,

$$\rho a_c \frac{\partial u}{\partial\xi} + \frac{\partial p}{\partial\xi} = 0, \quad \rho a_c \frac{\partial v}{\partial\xi} + \frac{\partial p}{\partial\eta} = 0, \quad \frac{\partial S}{\partial\xi} = 0 \tag{2}$$

The first equation of the system (1) yields, in this approximation. a relation which follows from those given above. The first and second equations of (2) together yield

$$\partial v / \partial\xi - \partial u / \partial\eta = 0 \tag{3}$$

and this shows that v is of the order of $u\lambda / l$. The first and third equations of (2) give the linearized Bernoulli integral

$$a^2 = a_0^2(\eta) - (\varkappa - 1) a_c u \tag{4}$$

We see that in the first approximation the streamlines coincide with the lines $\eta = \text{const.}$, and we note that

$$a_0^2 = a_c^2 + \eta\frac{da_0^2}{d\eta} = a_c^2 - 2a_c^2\frac{\eta}{R_r}, \quad \theta' = \frac{a_c}{R_c}$$

where R_c and R_r denote the radii of curvature of the caustic and of the rays, respectively. Using (4) and the last relation we obtain from (1) the following expressions:

$$\left(\frac{\varkappa+1}{a_c}u+\frac{\eta}{R}\right)\frac{\partial u}{\partial \xi}-\frac{\partial v}{\partial \eta}=0,\quad \frac{1}{R}=2\left(\frac{1}{R_c}-\frac{1}{R_r}\right) \tag{5}$$

Equations (3) and (5) together form a closed system. Let us rewrite them in the dimensionless variables

$$(u'+y)\frac{\partial u'}{\partial x}-\frac{\partial v'}{\partial y}=0,\quad \frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}=0 \tag{6}$$

$$u'=(\varkappa+1)\frac{u}{a_c}\frac{R}{l},\quad v'=(\varkappa+1)\frac{v}{a_c}\left(\frac{R}{l}\right)^{3/2} x=\xi/\lambda,\quad y=\eta/l$$

The boundary conditions for the system (6) are obtained from geometrical acoustics. At large negative y the perturbations u' and v' should tend to zero. At large positive y we have a signal arriving at the caustic in the form

$$u' = -\mu y^{-1/4} f\left(x+{}^2/_3 y^{3/2}\right) \tag{7}$$

The quantity μ characterizes the amplitude of the signal. Equations analogous to (6) were obtained in a slightly different manner in [4—6].

From the analytic point of view the system (6) appears to be complicated. We shall explain the qualitative aspects of the phenomenon using a linear formulation of the problem of reflection of a wave from the caustic. With this in mind, we introduce the velocity potential and instead of (6), we write

$$y\frac{\partial^2\Phi}{\partial x^2}-\frac{\partial^2\Phi}{\partial y^2}=0 \tag{8}$$

Following [7], we solve (8) using the method of Fourier transforms. This yields

$$\Phi=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega x}k(\omega)\,\mathrm{Ai}\left(-|\omega|^{2/3}y\right)d\omega \tag{9}$$

where $\mathrm{Ai}(t)$ is the Airy function decaying exponentially with $t\to+\infty$. The boundary condition (7) yields

$$k(\omega)=-\sqrt{2\pi}\,\mu|\omega|^{-5/6}(1+i\,\mathrm{sign}\,\omega)\int_{-\infty}^{+\infty}e^{i\omega p}f(p)\,dp$$

In addition, the asymptotic behavior given by (9) yields, at large y, the following expression for the reflected wave:

$$u=-\frac{\mu y^{-1/4}}{2\pi}\int_{-\infty}^{+\infty}e^{-i\omega q}\beta(\omega)\,d\omega \tag{10}$$

$$\beta(\omega)=-i\,\mathrm{sign}\,\omega\int_{-\infty}^{+\infty}e^{i\omega p}f(p)\,dp,\quad q=x-\frac{2}{3}y^{3/2}$$

Thus, using the formula (10) we can describe the reflected signal for any input signal (7). Let us investigate the structure of the reflected signal in the case of a shock wave arriving at the caustic, followed by the region of steady flow.
In this case we have

$$f(p) = \theta(p) - \theta(p - \delta)$$

where $\theta(p)$ is the Heaviside function and δ is a sufficiently large constant. A solution of (8) was given in [7] in terms of the hypergeometric functions. Fig. 2 depicts, with a dashed line, the relation $u(x)$ obtained from (9) when $y = 1$ for $\delta = 20$ and $\mu = 1$. Using the formula (10), we obtain the reflected signal in the form

$$u = \frac{\mu y^{-1/4}}{\pi} (\ln |q| - \ln |q - \delta|)$$

Indeed, the nonlinear terms will appear in the vicinity of the logarithmic singularity and the latter will be replaced, in the nonlinear solution, by a finite but quite significant increase in the signal's amplitude [8, 9]. The character of the solution however depends on the wavelength of the signal arriving at the caustic.

To explain the influence which a wave of finite length has on the structure of the reflected signal, we shall consider the incident wave the form of which is described by the formula

$$f(p) = \frac{p - \lambda}{\lambda} [\theta(p - \lambda) - \theta(p)]$$

In this case the reflected wave has the form

$$u = \frac{\mu y^{-1/4}}{\pi} \left(1 + \frac{q - \lambda}{\lambda} \ln \left| \frac{q - \lambda}{q} \right|\right)$$

Clearly, the length of the reflected wave will be of the same order as the length of the incident wave. For this reason the peak increase in the amplitude of the signal will be significantly narrower for a short length wave. Figure 2 shows for comparison the graphs of $u(x)$ with $y = 1$, for $\mu = 1, \lambda = 20$ (curve *1*) and for $\lambda = 1$ (curve *2*). It is clear that the narrow peak is more affected by the nonlinear effects than the wide peak. Therefore the increase of the signal amplitude in the nonlinear solution will be the smaller the shorter the wavelength of the signal arriving at the caustic.

Let us consider a case, which is of greatest practical interest, in which the signal arriving at the caustic has the form of the N-wave. In this case we have

$$f(p) = 2\frac{p}{\lambda}\left[\theta\left(p - \frac{\lambda}{2}\right) - \theta\left(p + \frac{\lambda}{2}\right)\right]$$

The reflected wave is described by the formula

$$u = 2\frac{\mu y^{-1/4}}{\pi}\left[1 + \frac{q}{\lambda}\left(\ln\left|q - \frac{\lambda}{2}\right| - \ln\left|q + \frac{\lambda}{2}\right|\right)\right]$$

The reflected wave has two rapidly decaying logarithmic singularities, both of the same sign. The meaning of the quantities introduced clearly implies that the wavelength of the signal arriving at the caustic must be of the order of unity when the dimensionless variables x and y are used. The relation $u(x)$ at $y = 1$ is shown for $\mu = 1, \lambda = 1$ in Fig. 2 (curve *3*). The peaks of the increased signal amplitude will be quite narrow, therefore the signal amplitude in the reflected and incident wave will be of the same order.

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